COUNTABLE MODELS OF $\aleph_1$-CATEGORICAL THEORIES

BY
MICHAEL MORLEY[1]

Countable models of $\aleph_1$-categorical theories are classified. It is shown that such a theory has only a countable number of nonisomorphic countable models.

A theory (formulated in the predicate calculus) is categorical in power κ (κ-categorical) if it has a model of power κ and any two such are isomorphic. In [3] I proved the conjecture of Łos' that a theory categorical in one uncountable power is necessarily categorical in every uncountable power. The example of the theory of algebraically closed fields of characteristic 0 shows that such a theory need not be $\aleph_0$-categorical. However, one might expect that the isomorphism types of countable models of such theories could be classified in some particularly neat fashion. In the example mentioned the isomorphism type can be characterized by the number of algebraically independent elements. Thus, it is possible to find an increasing sequence of length $\omega + 1$ of models:

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}_\omega$$

such that every countable model is isomorphic to some member of the sequence and, for each n, $\mathfrak{A}_{n+1}$ is the "next larger" model than $\mathfrak{A}_n$. The results of this paper, together with a more recent result of Marsh [2], show that such a sequence of models can be found for every theory which is $\aleph_1$- but not $\aleph_0$-categorical. In every known instance of such a theory no two members of this sequence are isomorphic. It is an open question whether this is true in general. Indeed, it is not known whether a theory which is $\aleph_1$- but not $\aleph_0$-categorial must have an infinite number of isomorphism types of countable models.

We have adopted the following compromise with respect to prerequisites. The definitions and statements of theorems assumes only a basic knowledge of model

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theory but an understanding of the proofs requires an understanding of the methods of [3]. Some of the results of this paper were announced in [4].

1. **Preliminaries.** A *relation system*, $\mathfrak{A}$, is a set A, the *universe* of $\mathfrak{A}$, together with an indexed set of finitary relations, finitary functions and distinguished elements. (We shall adopt the now common convention of denoting relation systems by Gothic letters, $\mathfrak{A}$, and their universe by the corresponding Roman letter A.) Two relation systems are *similar* if they have the same index set and corresponding relations and functions have the same degree. For similar systems one defines $\mathfrak{A}$ a *subsystem* of $\mathfrak{B}$ ($\mathfrak{A} \subseteq \mathfrak{B}$) and $\mathfrak{A}$ *isomorphic* to $\mathfrak{B}$ ($\mathfrak{A} \cong \mathfrak{B}$) in the obvious fashion. Corresponding to each similarity type of relation systems there is a first order with identity language having relation symbols, function symbols, and individual constants corresponding to the relations, functions and distinguished elements of the relation systems. We shall always assume that this language is countable. The reader is presumed familiar with the notions of *formula* and *sentence* in such a language and what it means for a sequence of elements of $\mathfrak{A}$ to *satisfy* a formula in $\mathfrak{A}$ or for a sentence to be *valid* in $\mathfrak{A}$. A relation system $\mathfrak{A}$ is a *model* of a set of sentences if each of the sentences is valid in $\mathfrak{A}$. A *complete* theory (in a given language) is a maximal consistent set of sentences in that language. For each relation system $\mathfrak{A}$ there is a complete theory, $Th(\mathfrak{A})$, consisting of all sentences valid in $\mathfrak{A}$. We write $\mathfrak{A} \equiv \mathfrak{B}$ to indicate $Th(\mathfrak{A}) = Th(\mathfrak{B})$.

If $\mathfrak{A}$ is a relation system and $X \subseteq A$ we denoted by $(\mathfrak{A}, x)_{x \in X}$ the new relation system formed by taking the elements of X as distinguished elements. The language corresponding to $(\mathfrak{A}, x)_{x \in X}$ differs from that corresponding to $\mathfrak{A}$ by the addition of new individual constants corresponding to the elements, of X. If the letter a denotes an element of X we shall denote the new individual constant by bold face $\boldsymbol{a}$.

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are similar relation systems, $X \subseteq A$, and f is a function defined on X into B. Then f is an *elementary* map if $(\mathfrak{A}, x)_{x \in X} \equiv (\mathfrak{B}, f(x))_{x \in X}$. In particular, if $\mathfrak{A} \subseteq \mathfrak{B}$ and the identity map on $\mathfrak{A}$ into $\mathfrak{B}$ is an elementary map then $\mathfrak{A}$ is an *elementary* subsystem of $\mathfrak{B}$ ($\mathfrak{A} \prec \mathfrak{B}$).

A system $\mathfrak{A}$ is *minimal* if it has no proper elementary subsystems. It is *prime* if $\mathfrak{A} \equiv \mathfrak{B}$ implies that there is an elementary map of $\mathfrak{A}$ into $\mathfrak{B}$. If $X \subseteq A$ we say $\mathfrak{A}$ is *prime over* X if $(\mathfrak{A}, x)_{x \in X}$ is prime.(2) It can be shown (cf. [7]) that if $\mathfrak{A}$ is prime then it is prime over every finite $X \subseteq A$. $\mathfrak{B}$ is a *minimal elementary extension* of $\mathfrak{A}$ if $\mathfrak{B}$ is a proper elementary extension of $\mathfrak{A}$ and for all $\mathfrak{C}$, $\mathfrak{A} \prec \mathfrak{C} \prec \mathfrak{B}$ implies $\mathfrak{A} = \mathfrak{C}$ or $\mathfrak{B} = \mathfrak{C}$. $\mathfrak{B}$ is a *prime elementary extension* of $\mathfrak{A}$ if $\mathfrak{B}$ is a proper elementary extension of $\mathfrak{A}$ and for all other proper elementary exten-

(2) Strictly speaking, this terminology is ambiguous since the notion depends not only on the set X but the formulas satisfied by X. In our usage, however, the meaning will always be clear from context. A similar remark applied to the notation $S(X)$ introduced below.

sions $\mathfrak{C} \succ \mathfrak{A}$ there is an elementary map of $\mathfrak{B}$ into $\mathfrak{C}$ which is the identity on $\mathfrak{A}$.

Suppose T is a complete theory in some language and F the set of formulas having no free variable other than some fixed one, say v_0. Let S be the set of subsets of F maximal consistent with T. S may be given a compact, totally-disconnected topology(3) by taking as a basis the sets:

$$\{p \in S;\ \phi \in p\} \qquad (\phi \in F).$$

If $\mathfrak{B}$ is a model of T and $b \in \mathfrak{B}$ then the set of formulas satisfied by b in $\mathfrak{B}$ is some $p \in S$. We say b *realizes* p or b is of *type* p. In the particular case where $T = Th((\mathfrak{A}, x)_{x \in X})$ we shall write S as $S(X)$ and if $b \in \mathfrak{B}$, $\mathfrak{B} \succ \mathfrak{A}$ we shall say b realizes $p \in S(X)$ if b realizes p in $(\mathfrak{B}, x)_{x \in X}$. A system $\mathfrak{A}$ is *saturated* if for every $X \subseteq A$ with the cardinality of X less than that of A, every $p \in S(X)$ is realized in $\mathfrak{A}$. It can be shown (see [4] or [5]) that if $\mathfrak{A} \equiv \mathfrak{B}$, $\mathfrak{A}$ and $\mathfrak{B}$ are of the same power and both are saturated then $\mathfrak{A} \cong \mathfrak{B}$.

2. The Elementary Tower.

LEMMA (4). *Suppose T is a complete theory, $\mathfrak{A}$ a countable model of T and p the only limit point in some neighborhood of $S(A)$. If there is a proper elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ in which p is not realized then there is an uncountable elementary extension of $\mathfrak{A}$ in which only a countable number of elements realize p.*

Proof. Suppose $\mathfrak{A}$, $\mathfrak{B}$ and p are as in the hypothesis of the lemma. Since $\mathfrak{A}$ is a model of T every isolated point of $S(A)$ must be realized in $\mathfrak{A}$ (cf. lemma 4.1 of [3]). From this it follows that every isolated point of $S(A)$ can be realised by only one point. Hence no isolated point of $S(A)$ is realized in $\mathfrak{B} - \mathfrak{A}$. Let ϕ be a formula defining a neighborhood of p in which p is the only limit point. Since p is not realized in $\mathfrak{B}$ and no isolated point is realized in $\mathfrak{B} - \mathfrak{A}$ it follows that no point in the neighborhood defined by ϕ is realized in $\mathfrak{B} - \mathfrak{A}$. Thus, ϕ is satisfied by no element of $\mathfrak{B} - \mathfrak{A}$. By Vaught's two-cardinal theorem [5] there is an uncountable model of $Th(\mathfrak{A}, a)_{a \in w}$ in which only a countable number of elements satisfy ϕ and hence *a fortiori* only a countable number realize p.

THEOREM 1. *A complete theory T is $\aleph_1$-categorical if and only if every countable model has a prime elementary extension.*

Proof. Suppose T is $\aleph_1$-categorical and $\mathfrak{A}$ is a countable model of T. Using the results of [3] we know that T is *totally transcendental* and therefore $S(A)$ is countable and must have a point p which is the only limit point in some

(3) This is no more than a thinly disguised version of the Stone representation theorem for Boolean algebras. For a more detailed discussion see [3].

(4) A stronger form of this lemma was announced as theorem 1 of [4]. However there was an error (pointed out to me by Charlotte Stark Chell) in my proof of that result.

neighborhood of $S(A)$. T is $\aleph_1$-categorical so by theorem 5.5 of [3] every uncountable model of T is saturated and hence every uncountable elementary extension of $\mathfrak{A}$ has an uncountable number of points realizing p, From the preceding lemma it follows that every proper elementary extension of $\mathfrak{A}$ has an element realizing p. Let b be such an element in some elementary extension of $\mathfrak{A}$. Then for every proper elementary extension $\mathfrak{C} \succ \mathfrak{A}$ there is an elementary map of $A \bigcup \{b\}$ into C which is the identity on A. But by theorem 4.3 of [3] $A \bigcup \{b\}$ has a model of T prime over it. This model is the desired prime elementary extension of $\mathfrak{A}$.

Conversely, suppose every countable model of T has a prime proper elementary extension. Using the axiom of choice we choose for each isomorphism type of countable models a type of prime elementary extension. (We have not assumed that the prime elementary extension is unique, but see Theorem 2 below.) For each countable model $\mathfrak{A}$ then we define an increasing sequence of models $\{\mathfrak{A}_\alpha;\ \alpha \leqq \omega_1\}$ by $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{A}_{\alpha+1}$ is the prime elementary extension of $\mathfrak{A}_\alpha$ and for limit ordinals δ,

$$\mathfrak{A}_\delta = \bigcup_{\alpha = \delta} \mathfrak{A}_\alpha .$$

Let us call this the $\mathfrak{A}$-*tower*. Notice that every member of this sequence is countable except the last one, $\mathfrak{A}_{\omega_1}$, which has power $\aleph_1$.

Suppose $\mathfrak{B}$ is a countable elementary extension of $\mathfrak{A}$. Letting f_0 be the identity map of $\mathfrak{A}$ into $\mathfrak{B}$ we may proceed inductivity to extend this to elementary maps $f_\alpha ; \mathfrak{A}_\alpha \to \mathfrak{B}$ since each $\mathfrak{A}_{\alpha+1}$ is a prime elementary extension of $\mathfrak{A}_\alpha$. The induction will cease exactly at that α where f_α maps $\mathfrak{A}_\alpha$ *onto* $\mathfrak{B}$. This must occur for some $\alpha < \omega_1$ for, otherwise, we would have an elementary map of $\mathfrak{A}_{\omega_1}$ into $\mathfrak{B}$ which is impossible since $\mathfrak{A}_{\omega_1}$ is not countable Similarly, suppose $\{\mathfrak{B}_\alpha ; \alpha < \gamma\}$ is an increasing elementary chain of models with $\mathfrak{A} \prec \mathfrak{B}_0$. Then the same argument permits us to find a subsequence of the $\mathfrak{A}$-tower isomorphic to the increasing chain of $\mathfrak{B}$'s.

Next, suppose $\mathfrak{B} \succ \mathfrak{A}$ and $\mathfrak{B}$ has power $\aleph_1$. $\mathfrak{B}$ is equal to the union of an increasing chain of countable models $\{\mathfrak{B}_\alpha ; \alpha > \omega_1\}$; so it is isomorphic to the union of a subsequence of length ω_1 of the $\mathfrak{A}$-tower. That is, $\mathfrak{B}$ is isomorphic to $\mathfrak{A}_{\omega_1}$.

Finally, suppose $\mathfrak{B}$ and $\mathfrak{B}'$ are two models of T of power $\aleph_1$. From the results of the preceding paragraph it follows that there must be countable models $\mathfrak{A}$ and $\mathfrak{A}'$ such that $\mathfrak{B}$ and $\mathfrak{B}'$ are isomorphic to the last member of the $\mathfrak{A}$-tower and $\mathfrak{A}'$-tower respectively. There exists a countable model $\mathfrak{C}$ such that both $\mathfrak{A}$ and $\mathfrak{A}'$ may be mapped by elementary maps into $\mathfrak{C}$ (cf. lemma 1.2 of [5]). So isomorphic images of $\mathfrak{C}$ must appear in both the $\mathfrak{A}$-tower and $\mathfrak{C}'$-tower. Then $\mathfrak{B}$ and $\mathfrak{B}'$ are both elementary extensions of isomorphic images of $\mathfrak{C}$ so they are both isomorphic to the last member of the $\mathfrak{C}$-tower. This shows that T is $\aleph_1$-categorical and Theorem 1 is proved.

THEOREM 2. *Suppose T is $\aleph_1$categorical and $\mathfrak{A}$ a countable model of T.*

Then every prime elementary extension of $\mathfrak{A}$ is also a minimal elementary extension of $\mathfrak{A}$ and a prime elementary extension is therefore unique up to an isomorphism leaving $\mathfrak{A}$ fixed.

Proof. Suppose $\mathfrak{A}$ had a minimal elementary extension $\mathfrak{C}$ and $\mathfrak{B}$ was a prime elementary extension of $\mathfrak{A}$. By definition there would be an elementary map of $\mathfrak{B}$ into $\mathfrak{C}$ which was the identity on $\mathfrak{A}$. But this map must be *onto* $\mathfrak{C}$ for otherwise $\mathfrak{C}$ would not be a minimal extension. Thus, in order to prove the theorem it will suffice to show that $\mathfrak{A}$ has a minimal elementary extension. Consider the prime elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ constructed in the proof of Theorem 1. Suppose it were not a minimal elementary extension. Then there would be some model "interpolated" between $\mathfrak{A}$ and $\mathfrak{B}$. But since $\mathfrak{B}$ is a prime extension of $\mathfrak{A}$ there would be an elementary map f of $\mathfrak{B}$ into this interpolated model such that f is the identity on $\mathfrak{A}$. *A fortiori*, the image of f is not all of $\mathfrak{B}$. To complete the proof we shall show that the assumption that such an f exists leads to a contradiction. The argument is somewhat lengthy.

So, as in the proof of Theorem 1, assume that $\mathfrak{B} \succ \mathfrak{A}$ are countable models of $\aleph_1$-categorical theory T, p is the only limit point in some neighborhood of $S(A)$, $b \in \mathfrak{B}$ realizes p, and $\mathfrak{B}$ is a model prime over $X = A \bigcup \{b\}$, f is an elementary map of $\mathfrak{B}$ into a proper subsystem of itself and f is the identity on $\mathfrak{A}$. Let $\mathfrak{B}' = (\mathfrak{B}, x)_{x \in X}$ and $T' = Th(\mathfrak{B}')$. The theory T' is $\aleph_1$-categorical since X is countable and every model of T of power $\aleph_1$ is saturated. The theory T' is not $\aleph_0$-categorical since by a result of Ryll-Nardzewski [6] no $\aleph_0$-categorical theory can have an infinite set of distinguished elements. By definition $\mathfrak{B}'$ is a prime model of T'. A theorem of Vaught [7] says that a prime model of an $\aleph_1$-categorical but not $\aleph_0$-categorical theory is also a minimal model so $\mathfrak{B}'$ is a minimal model of T'. Denote by $\mathfrak{C}$ the image of $\mathfrak{B}$ under f. I assert that $b \notin \mathfrak{C}$. For if $b \in \mathfrak{C}$, then $X \subseteq C$ and $\mathfrak{C}' = (\mathfrak{C}, x)_{x \in X}$ would be a proper elementary subsystem of $\mathfrak{B}'$.

In particular, letting $f(b) = c$ we have $c \neq b$. Since f is the identity on A, c realizes the same point $p \in S(A)$ as b does. Indeed $\mathfrak{C}$ is the prime and minimal model of $T'' = Th((\mathfrak{B}, x)_{x \in Y})$. where $Y = A \bigcup \{c\}$. Since $x \in \mathfrak{B}$ and $\mathfrak{B}$ is prime over X, c must realize an isolated point is $S(X)$. That is the formulas satisfied by c in $\mathfrak{B}'$ are exactly the formulas of a principle dual ideal determined by a formula $\psi(v_0)$. Of course, ψ is a formula in the language of the theory T'; but this language differs from the language of the theory T only in the addition of new individual constants. Thus there is no loss of generality in assuming there is some finite sequence $a_2, \cdots, a_n$ in A such that $\psi(v_0) \equiv \psi(v_0, \mathbf{b}, \mathbf{a}_2, \cdots, \mathbf{a}_n)$ where $\psi(v_0, \cdots, v_n)$ is a formula in the language of the theory T. For typographical convenience we shall henceforth show only the first two arguments of ψ explicitly, viz. $\psi(v_0, v_1). \equiv \psi(v_0, v_1,. \mathbf{a}_2, \cdots, \mathbf{a}_n)$.

I assert that if $d \in \mathfrak{C}$ and satisfies in $\mathfrak{C}$ the formula $\psi(\mathbf{c}, v_0)$ then d does not realize the same point $p \in S(A)$ which is realized by b. For suppose it did. Then the mapping of $A \bigcup \{b, c\}$ onto $A \bigcup \{d, c\}$ which is the identity on $A \bigcup \{c\}$ would be an elementary map. Since $\mathfrak{B}$ is prime over $A \bigcup \{b\}$ it is also prime over $A \bigcup \{b, c\}$, so there would be an elementary map $g : \mathfrak{B} \to \mathfrak{C}$ which is the identity on $A \bigcup \{c\}$. But g maps $\mathfrak{C}$ onto a proper subsystem of itself and so the same argument given above to show that $f(b) \neq b$ shows that $g(c) \neq c$, a contradiction.

On the other hand, c does realize the point p in $S(A)$ and hence c has all the same first order properties with respect to $\mathfrak{A}$ that b does. In particular, any element satisfying $\psi(v_0, \mathbf{c})$ must realize p. Combining this with the result of the last paragraph we have the following anti-symmetry property, $\chi(v_0)$, satisfied by c;

$$\chi(v_0) \equiv (v_1)(\psi(v_0, v_1) \to \neg\psi(v_1, v_0)).$$

Let U be some neighborhood of p in $S(A)$ which contains no other limit point. Then there is some formula $\rho(v_0)$ such that U consists of all prime dual ideals containing ρ. (Of course, ρ may involve individual constants corresponding to the elements of $\mathfrak{A}$). The formula:

$$\chi(v_0) \,\&\, \rho(v_0) \,\&\, \psi(\mathbf{c}, v_0)$$

is satisfied by b so, since $\mathfrak{C} \prec \mathfrak{B}$; it must be satisfied by some element, say d_0, in $\mathfrak{C}$. We have already shown that this implies that d_0 does not realize the point $p \in S(A)$. Therefore d_0 must realize an isolated point of $S(A)$ and as mentioned in the proof of Lemma 1, therefore $d_0 \in \mathfrak{A}$. Then the formula $\psi(v_0, \mathbf{d}_0)$ determines a neighborhood of p in $S(A)$. Repeating the above argument we can find a $d_1 \in \mathfrak{A}$ which satisfies:

$$\chi(v_0) \,\&\, \rho(v_0) \,\&\, \psi(v_0, \mathbf{d}_0) \,\&\, \psi(\mathbf{c}, v_0).$$

Proceeding inductively we may find a sequence $\{d_n ; \in \omega\}$ of elements of A such that $m > n$ implies $\psi(\mathbf{d}_m, \mathbf{d}_n)$ and $\chi(\mathbf{d}_n)$. This says that ψ determines a linear ordering of the d_n's which contradicts theorem 3.9 of [3]. Theorem 2 is now proved.

3. **The number of countable models.** Suppose T is $\aleph_1$-categorical. Vaught has proved that T must have a prime model, $\mathfrak{A}$. As in the proof of Theorem 1 we may construct an $\mathfrak{A}$-*tower* $\{\mathfrak{A}_\alpha ; \alpha < \omega_1\}$ such that $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{A}_{\alpha+1}$ is the prime elementary extension of $\mathfrak{A}_\alpha$ and for non-zero limit ordinals δ, $\mathfrak{A}_\delta = \bigcup_{\alpha<\delta} \mathfrak{A}_\alpha$. Since $\mathfrak{A}$ is a prime model the arguments used to prove Theorem 1 shows that every countable model of T is isomorphic to some member of this tower. It remains to consider which members of the tower are isomorphic. One possibility is that T is $\aleph_0$-categorical so that all members of the tower are isomorphic. Let us exclude this case and suppose that T is $\aleph_1$-categorical but not $\aleph_0$-categorical. Vaught [7] has shown that under this hypothesis the prime model is also minimal. Hence $\mathfrak{A}_0$ is not isomorphic to $\mathfrak{A}_\alpha$ for any $\alpha > 0$. Vaught [7] has also shown that no complete theory T can have exactly two isomorphism types of countable models. In doing so he proves that: if T is not $\aleph_0$-categorical then no model

prime over a finite set is saturated. From this it follows that in our tower of models, $\mathfrak{A}_n$ is not saturated for any $n \in \omega$. On the other hand, Vaught has also shown that there must be a countable saturated model of T. So there is a countable $\alpha \geqq \omega$ such that $\mathfrak{A}_\alpha$ is saturated. Since the saturated model is isomorphic to a proper elementary subsystem of itself there is a countable β such that $\mathfrak{A}_\alpha \cong \mathfrak{A}_{\alpha+\beta}$. By induction therefore $\mathfrak{A}_{\alpha+\gamma} \cong \mathfrak{A}_{\alpha+\beta+\gamma}$ for all countable γ. In particular, I assert that $\mathfrak{A}_\alpha \cong \mathfrak{A}_{\alpha+\beta\tau}$ for all countable τ. The proof is by induction on τ. The only difficulty in the induction occurs at limit ordinals; there we use the fact [5] that the union of a countable elementary chain of countable saturated models is a countable saturated model. Thus from α on the tower repeats isomorphism types with period β. We have proved:

THEOREM 3. *If T is $\aleph_1$-categorical but not $\aleph_0$-categorical then the number of isomorphism types of countable models is countable, i.e., finite or denumerably infinite.*

There are no examples known where the number of isomorphism types is finite, but whether any such exist is an open question.

We may sharpen the results of the last theorem by:

THEOREM 4. *If T is an $\aleph_1$- but not $\aleph_0$-categorical theory and $\{\mathfrak{A}_\alpha; \alpha < \omega_1\}$ the tower described above, then $\mathfrak{A}_\delta$ is saturated for every limit ordinal $\delta > 0$.*

Proof. We shall prove something slightly stronger, namely, if δ is a limit ordinal and $\alpha < \delta$ then every point of $S(A_\alpha)$ is realized in $\mathfrak{A}_\delta$. To do this is it sufficient to prove the following: If $\{\mathfrak{B}_n; n \in \omega\}$ is an increasing elementary chain of countable models of T then every point of $S(B_0)$ is realized in $\bigcup_{n \in \omega} \mathfrak{B}_n$. From [3] we know that $S(B_0)$ is countable and hence must contain a point p_0 which is the only limit point in some neighborhood. This point must be realized in $\mathfrak{B}_1$. As discussed in [3] the identity map i: $\mathfrak{B}_0 \to \mathfrak{B}_1$ induces a continuous, onto projection $i^*: S(B_1) \to S(B_0)$. Since $S(B_1)$ is countable and compact $i^{*-1}(p_0)$ must contain some point p_1, which is the only limit point in some neighborhood of $S(B_1)$. Then there must be some element of $\mathfrak{B}_2$ which realizes p_1 and therefore also realizes p_0. Proceeding inductively we may find for each $n \in \omega$ a $p_n \in S(B_n)$ and an element $b_n \in \mathfrak{B}_{n+1}$ such that b_n realizes p_n and p_{n+1} projects onto p_n. By Lemma 4.4(a)(i) of [3] there is an $n_0 \in \omega$ such that all the p_n's with $n > n_0$ have the same transcendental *degree* and *rank*. It is shown in the proof of theorem 4.6 of [3] that this implies that the corresponding b_n's are *indiscernible* over B_0. Thus, $\bigcup_{n \in \omega} \mathfrak{B}_n$ is a model of T containing an infinite set of indiscernibles over B_0. It is shown in Theorem 5.4 of [3] that under these circumstances if some $q \in S(B_0)$ were not realized in $\bigcup_{n \in \omega} \mathfrak{B}_n$ then T would have an uncountable model which is not saturated. But this is impossible by Theorem 5.5.

While this paper was in preparation the following theorem was proved by Marsh [2].

THEOREM. *If T is $\aleph_1$-categorical then every elementary extension of a saturated model is saturated.*

Combining this with Theorem 4 we see that in our tower of countable models they are all saturated and hence isomorphic to each other from the ωth step on. The remaining open question is whether the first ω models of the tower must all be distinct isomorphism types. In every known example of $\aleph_1$- but not $\aleph_0$-categorical theories they are distinct.

Suppose T is an $\aleph_1$- but not $\aleph_0$-categorical theory in a countable language L, L' a language which extends L by the addition of a countable number of individual constants and T' a complete extension of T in L'. Then T' is also $\aleph_1$- but not $\aleph_0$-categorical. Marsh [2] has proved the following theorem,

THEOREM. *If T is $\aleph_1$- but not $\aleph_0$-categorical then there is a complete extension T' of T in a language which extends the language of T by only a finite number of new individual constants such that in the tower of countable models of T' the first ω models are of distinct isomorphism types.*

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UNIVERSITY OF WISCONSIN,
MADISON, WISCONSIN